

Modelos de Colas Exponenciales

1. Introducción a las Colas Exponenciales: M/M

a) Procesos Markov nacimiento (llegada) y muerte (salida).

Estado $\equiv N(t)$

b) E.Equilibrio: $\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1, \dots, \lambda_0}}{\mu_n, \dots, \mu_1}}$,
 $\pi_n = \pi_0 \frac{\lambda_{n-1, \dots, \lambda_0}}{\mu_n, \dots, \mu_1}, n = 1, 2, 3, \dots$

c) C.Equilibrio: $S_1 = 1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1, \dots, \lambda_0}}{\mu_n, \dots, \mu_1} = 1/\pi_0 < \infty$,

$S_2 = \sum_{n=0}^{\infty} (1/\lambda_n \prod_{i=0}^{n-1} \lambda_i / \mu_{i+1}) \infty \rightarrow \exists n_0, \forall n > n_0, \lambda_n / \mu_n < 1.$

2. Modelo M/M/1: un Servidor

a) $\lambda_n = \lambda, n=0,1, \dots, \mu_n = \mu, n = 1, 2, \dots$, diagrama de transición

b) EE: $\mu\pi_1 = \lambda\pi_0, \lambda\mu_{n-1} + \mu\pi_{n+1} = (\lambda + \mu)\pi_n, n \geq 1$

c) C.E: $S_1 = \sum_{n=0}^{\infty} \rho^n < \infty, S_1 = 1/\lambda \sum_{n=0}^{\infty} 1/\rho^n = \infty$,
 $\rho = \frac{\lambda}{\mu} < 1 \rightarrow M/M/1$ ergódico

d) $\pi_0 = 1/S_1 = 1 - \rho, \pi_n = (1 - \rho)\rho^n \sim G(1 - \rho)$,

Servidor.ocupado $P(N = 1) = \rho, P(N \geq n) = \rho^n$

e) $L = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}, \sigma_N^2 = \frac{\rho}{(1 - \rho)^2}, L'_q = \frac{1}{1 - \rho} = L_q / \rho^2$,
 $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{(1 - \rho)} = L - (1 - \pi_0), \frac{L_q}{\lambda} = \frac{L}{\mu}$

f) $W = E(w) = E(s)/(1 - \rho), E(s) = \frac{1}{\mu}$,
 $W_q = E(q) = \rho E(s)/(1 - \rho), E(q|q > 0) = W$,
 $F_q(t) = 1 - \rho e^{-t/W}, t \geq 0, F_w(t) = 1 - e^{-t/W}, t \geq 0$

3. Modelos M/M/1/K: Capacidad K Finita del Sistema

a) $\lambda_n = \{\lambda, n = 0, 1, \dots, K - 1; 0, n \geq K\}$,

$\mu_n = \{\mu, n = 1, 2, \dots, K; 0, n > K\}$

b) EE: $\mu\pi_1 = \lambda\pi_0, \lambda\pi_{n-1} + \mu\pi_{n+1} = (\lambda + \mu)\pi_n, \lambda\pi_{K-1} = \mu\pi_K, \sum_{n=0}^K \pi_n = 1$

c) $\pi_n = (\lambda/\mu)^n \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{K+1}}, \lambda \neq \mu, n = 0, 1, \dots, K$,
 $\pi_n = \frac{1}{K+1}, \lambda = \mu, n = 0, 1, \dots, K$

d) $L = \frac{\lambda(1 - (K+1)(\lambda/\mu)^K + K(\lambda/\mu)^{K+1})}{(\mu - \lambda)(1 - (\lambda/\mu)^{K+1})}, \lambda \neq \mu; L = K/2, \lambda = \mu$

e) $L_s = 1 - \pi_0, L_q = L - (1 - \pi_0), \lambda_e = \lambda(1 - \pi_K), \rho = \lambda_e W = 1 - \pi_0$

f) $W = L/\lambda_e, W_q = L_q/\lambda_e, E(q|q > 0) = W_q/\rho$;
 $q_n = (N_e = n) = \pi_n/(1 - \pi_K), n=0,1, \dots, K-1$
 $F_w(t) = 1 - \sum_{n=0}^{K-1} q_n F_{P(\mu t)}(n), F_q(t) = 1 - \sum_{n=0}^{K-2} q_{n+1} F_{P(\mu t)}(n),$
 $F_{P(\mu t)}(n) = \sum_{i=0}^n \frac{(\mu t)^i e^{-\mu t}}{i!} = \int_t^{\infty} \mu \frac{\mu^n x^n e^{-\mu x}}{n!} dx$

4. Modelo M/M/c: c Servidores Paralelos

a) $\lambda_n = \lambda, n = 0, 1, \dots, \mu_n = n\mu, n = 1, 2, \dots; c; \mu_n = c\mu, n \geq c$

b) EE: $\mu\pi_1 = \lambda\pi_0; \lambda\pi_{n-1} + (n + 1\mu\pi_{n+1}) = (\lambda + n\mu)\pi_n, 1 \leq n \leq c - 1$;

$\lambda\pi_{n-1} + c\mu\pi_{n+1} = (\lambda + c\mu)\pi_n, n \geq c$

c) $\pi_0 = [\sum_{n=0}^{c-1} \frac{r^n}{n!} + \frac{r^c}{c!(1-\rho)}]^{-1}, r = \frac{\lambda}{\mu}, \rho = r/c$;

$\pi_n = \pi_0 r^n / n!, n = 0, 1, \dots, c; \pi_n = \pi_0 r^n / (c! c^{n-c}), n \geq c$

d) $C(c, r) = P(N \geq c) = \pi_0 r^c / (c!(1 - \rho))$

e) $L_q = C(c, r) \frac{\rho}{1 - \rho}, W_q = \frac{C(c, r)}{c\mu(1 - \rho)}, E(q|q > 0) = \frac{1}{c\mu(1 - \rho)}$,
 $W = W_q + W_s = 1/\mu(1 + \frac{C(c, r)}{c(1 - \rho)}), L = \lambda W = c\rho + C(c, r) \frac{\rho}{1 - \rho}$

f) $F_q(t) = 1 - C(c, r)e^{-(1-\rho)c\mu t}; q_l = \frac{1}{c\mu(1-\rho)} \ln(\frac{C(c, r)}{1-l})$;
 $F_w(t) = 1 + \frac{r - c + 1 - C(c, r)}{c - 1 - r} e^{-\mu t} + \frac{C(c, r)}{c - 1 - r} e^{-(1-\rho)c\mu t}, r \neq c - 1$;
 $1 - [1 + C(c, r)\mu t]e^{-\mu t}, r = c - 1$

g) Dos servidores: $\pi_0 = \frac{1-\rho}{1+\rho}, \pi_n = 2\rho^n \pi_0, C(2, r) = \frac{2\rho^2}{1+\rho}$,
 $L_q = \frac{2\rho^3}{1-\rho^2}, W_q = \frac{\rho^2 E[s]}{1-\rho^2}, L = \frac{2\rho}{1-\rho^2}, W = \frac{E[s]}{1-\rho^2}$

5. Modelo M/M/∞: Infinitos Servidores

a) $\lambda_n = \lambda, n = 0, 1, 2, \dots, \mu_n = n\mu, n = 0, 1, 2, \dots$

b) EE: $\mu\pi_1 = \lambda\pi_0; \lambda\pi_{n-1} + (n + 1)\mu\pi_{n+1} = (\lambda + n\mu)\pi_n, n \geq 1$

c) $N \sim P(r = \lambda/\mu), \pi_n = \frac{(\lambda/\mu)^n e^{-(\lambda/\mu)}}{n!}, n = 0, 1, \dots$

d) $L = \lambda/\mu, \sigma_N^2 = \lambda/\mu, L_q = W_q = 0$,
 $W = W_s = 1/\mu, w \sim s \sim \text{Exp}(\mu)$

6. Modelo $M/M/1/K/K$: Población Finita

- a) $\lambda_n = (K - n)\lambda, n = 0, 1, \dots, K - 1$ y 0 el resto; $\mu_n = \mu, n = 0, 1, \dots, K - 1$ y 0 el resto
- b) EE: $\mu\pi_1 = K\lambda\pi_0; (K - n + 1)\lambda\pi_{n-1} + (n + 1)\mu\pi_{n+1} = ((K - n)\lambda + \mu)\pi_n, 1 \leq n \leq K - 1; \lambda\pi_{K-1} = \mu\pi_K$
- c) $\pi_0 = \left(\sum_{n=0}^K \frac{K!}{(K-n)!} \left(\frac{\lambda}{\mu} \right)^n \right)^{-1} = B(K, z), z = \frac{\mu}{\lambda}, \pi_n = \frac{K!}{(K-n)!} \left(\frac{\lambda}{\mu} \right)^n \pi_0$

- d) $\rho = 1 - \pi_0 = 1 - B(K, z); \lambda_e = \mu\rho = \lambda(K - L); L_q = \lambda_e W_q; L = K - \frac{\lambda_e}{\lambda}$
- e) $W = \frac{K}{\lambda_e} - \frac{1}{\lambda}; W_q = \frac{K}{\lambda_e} - \frac{1}{\lambda} - \frac{1}{\mu}; \frac{L}{K-L} = W\lambda; E[q|q > 0] = \frac{W_q}{\rho}$
- f) $F_q(t) = 1 - \sum_{n=0}^{K-1} q_{n+c} F_{P(\mu t)}(t), q_n = \frac{(K-n)\pi_n}{K-L}, n = 0, 1, \dots, K - 1$
- g) P(una máquina averiada) = $\frac{W}{W+E(O)}, E(O) = 1/\lambda$